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Tau-wavelets of Haar

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Abstract. In this paper we construct a new type of Haar wavelets, called τ-wavelets of Haar, using the arithmetics of the solutions $\tau = \frac{1}{2}(1+\sqrt{5})$ and $\sigma = \frac{1}{2}(1-\sqrt{5})$ of the algebraic equation $x^2 = x + 1$.

1. Introduction

Most of the discrete (orthogonal) wavelet analysis is based on binary subdivisions of intervals. This is the key point of the multiresolution analysis [Me92, Me93, D92]. An increasing sequence $\{V_j\}_{j\in\mathbb{Z}}$ of subsets of $L^2(\mathbb{R})$ is defined by

$$f(x) \in V_j \Longleftrightarrow f(2x) \in V_{j+1}. \tag{1.1}$$

The following algebraic identity is crucial in the subdivision process

$$\frac{1}{2^j} = \frac{1}{2^{j+1}} + \frac{1}{2^{j+1}}. \tag{1.2}$$

Once a 'father wavelet' $\varphi(x) \in V_0$ has been identified, a Fourier algorithm allows one to find a second function, the 'mother wavelet', in the orthogonal complement W_0 of V_0 in V_1, i.e. $V_1 = V_0 \oplus W_0$.

Defining W_j, $j \in \mathbb{Z}$, by

$$f(x) \in W_0 \Longleftrightarrow f(2^j x) \in W_j \tag{1.3}$$

then

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) \qquad k \in \mathbb{Z} \tag{1.4}$$

is an orthonormal basis of W_j, and we have

$$L^2(\mathbb{R}) = \bigoplus_{-\infty}^{+\infty} W_j. \tag{1.5}$$

For example, the Haar basis is recovered by just choosing $\varphi(x) = \chi_{[0,1]}(x)$.

The aim of the present paper is to parallel on an elementary level this construction of the discrete set $\{\psi_{j,k}\}$ by considering a subdivision process based on the number $\tau = \frac{1}{2}(1+\sqrt{5}) \approx 1.618\ldots$ (the 'golden mean'). The counterpart of (1.2) is the following relation:

$$\frac{1}{\tau^j} = \frac{1}{\tau^{j+1}} + \frac{1}{\tau^{j+2}}. \tag{1.6}$$

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From that point of view, τ is the closest relative of 2, in spite of the fact that it is the furthest irrational in terms of approximation by rationals!

In that lies an obvious motivation for a development of a multiresolution analysis based on the number τ. Here we construct the Haar basis, called Haar τ-wavelet basis, as a preliminary to any deeper multiresolution theory.

From a physicist's viewpoint, once the τ-wavelets are identified and constructed, one can expect to find them particularly advantageous in problems where the pentagonal symmetry, combined with self-similarity, plays a privileged role, for example in problems involving physical spectra of quasicrystalline solids. There it is quite natural to use them as a tool complementary to the Fourier exponential in considering the quantum mechanical problems of energy spectra, integrated densities of states or wavefunctions for transport electrons interacting with atoms situated in vertices of a quasilattice with local icosahedral symmetry [JM95]. Indeed as exponential functions of Fourier adequately reflect all lattice geometrical, optical and quantum features (translational and possibly rotational invariance, diffusion pattern, Bloch waves) 'quasilattice' functions are still lacking. Those objects should reflect on their side the quasilattice spatial organization based on invariance under dilation and, possibly, non-global rotation [BDG94].

For simplicity we consider here only one-dimensional problems. It should be noted that two- and three-dimensional cases differ mainly in the scale of technical computations.

2. Arithmetics of τ-adics

Let us recall that the algebraic equation

$$x^2 = x + 1 \tag{2.1}$$

has the solutions

$$\tau = \tfrac{1}{2}(1+\sqrt{5}) \qquad \text{and} \qquad \sigma = \tfrac{1}{2}(1-\sqrt{5}) \tag{2.2}$$

which satisfy the identities

$$\tau^j = \tau^{j-1} + \tau^{j-2} \qquad \text{and} \qquad \sigma^j = \sigma^{j-1} + \sigma^{j-2} \qquad \text{for any } j \in \mathbb{Z} \tag{2.3}$$

and also

$$\tau\sigma = -1 \qquad \tau + \sigma = 1 \tag{2.4}$$

$$\tau^j = F_j\tau + F_{j-1} \qquad \sigma^j = F_j\sigma + F_{j-1} \tag{2.5}$$

where F_j, $j \in \mathbb{Z}_{\geqslant -1}$ are the terms of the Fibonacci series

$$F_{j+1} = F_j + F_{j-1} \tag{2.6}$$

with

$$F_{-1} = 1 \qquad F_0 = 0 \qquad F_1 = 1 \qquad F_2 = 1 \qquad F_3 = 2$$
$$F_4 = 3 \qquad F_5 = 5 \qquad F_6 = 8 \qquad F_7 = 13, \ldots .$$

The property (2.3) is essential for the arithmetics based on the number τ.

Let x be a real positive number in the interval $]0, 1[$. Let $m_0 = m_0(x)$ be the smallest natural number such that

$$\frac{1}{\tau^{m_0}} \leqslant x < \frac{1}{\tau^{m_0-1}}. \tag{2.7}$$

Then we have

$$0 \leqslant x - \frac{1}{\tau^{m_0}} < \frac{1}{\tau^{m_0-1}} - \frac{1}{\tau^{m_0}} = \frac{1}{\tau^{m_0+1}}.$$

Next let m_1 be the smallest natural number such that

$$\frac{1}{\tau^{m_1}} \leqslant x - \frac{1}{\tau^{m_0}} < \frac{1}{\tau^{m_1-1}} \leqslant \frac{1}{\tau^{m_0+1}} \tag{2.8}$$

or equivalently

$$0 \leqslant x - \frac{1}{\tau^{m_0}} - \frac{1}{\tau^{m_1}} < \frac{1}{\tau^{m_1+1}}.$$

In this way one can construct an increasing series

$$\frac{1}{\tau^{m_0}}, \frac{1}{\tau^{m_0}} + \frac{1}{\tau^{m_1}}, \ldots \tag{2.9}$$

evidently converging to x. Since $\frac{1}{\tau^{m_1}} < \frac{1}{\tau^{m_0+1}}$ follows from (2.8), we have $m_1 \geqslant m_0 + 2$. Hence the successive powers of τ appearing in each term of the series (2.9) differ at least by 2.

Similar reasoning with powers of τ, first the positive ones and then the negative ones, allows one to express any positive number x as the limit of the increasing series

$$\tau^{n_0}, \tau^{n_0} + \tau^{n_1}, \ldots, \tau^{n_0} + \tau^{n_1} + \cdots + \frac{1}{\tau^{m_0}} + \frac{1}{\tau^{m_1}} + \cdots$$

where again the difference of two successive exponents of τ is at least 2. Consequently any real number x can be uniquely presented in the form

$$\begin{aligned} & x = \tau^{n_0} \sum_{n=0}^{\infty} k_n \tau^{-n} \qquad n_0 \in \mathbb{Z} \\ & k_0 = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} \\ & k_n = \begin{cases} 0 \text{ or } 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ 0 \text{ or } -1 & \text{if } x < 0 \end{cases} \qquad \text{for all } n > 0,\ k_n k_{n+1} = 0. \end{aligned} \tag{2.10}$$

We say that x is τ-*integer* precisely if, in the form (2.10), it is written with no negative powers of τ, i.e. $k_n = 0$ for all $n_0 - n < 0$. For example, a few of the first τ-natural numbers are the following:

$$0, 1, \tau, \tau^2, \tau^2 + 1, \tau^3, \tau^3 + 1, \tau^3 + \tau, \tau^4, \tau^4 + 1, \tau^4 + \tau, \tau^4 + \tau^2, \tau^4 + \tau^2 + 1, \ldots. \tag{2.11}$$

We say that a number x is τ-*rational* if the sum in (2.10) is finite,

$$k_n = 0 \qquad \text{for all } n > n_{\min} \in \mathbb{Z}.$$

Examples of τ-rational numbers are

$$\sqrt{5} = \tau + \frac{1}{\tau} \qquad \text{and} \qquad 2 = \tau + \frac{1}{\tau^2} \tag{2.12}$$

as well as all the integers and τ-integers and all the elements of the quadratic ring $\mathbb{Z}[1, \tau] \equiv \mathbb{Z}[\tau]$ of τ-integers. In view of the identities (2.3), we can write the latter as

$$\mathbb{Z}[\tau] = \{x = m + n\sqrt{5} | m, n \in \mathbb{Z}\}. \tag{2.13}$$

Note, however, that numbers 'simple' as $\frac{1}{2}$ are represented as an infinite series:

$$\frac{1}{2} = \sum_{j=0}^{\infty} \frac{1}{\tau^{3j+1}}. \tag{2.14}$$

3. τ-wavelets of Haar

Consider the Hilbert space $L^2[0,1]$ and let us recall its well known basis of Haar. Let $h(x)$ be the function defined as follows:

$$h(x) = \begin{cases} 1 & \text{for } x \in [0, \frac{1}{2}] \\ -1 & \text{for } x \in]\frac{1}{2}, 1] \\ 0 & \text{for } x \notin [0, 1]. \end{cases} \tag{3.1}$$

Then one defines the sequence of the functions

$$h_{j,k}(x) = 2^{j/2} h(2^j x - k) \tag{3.2}$$

for all $j \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $k < 2^j$, that is, k takes all the integer values in the interval between 0 and $2^j - 1$.

The support of $h_{j,k}(x)$ is the diadic interval $[\frac{k}{2^j}, \frac{k+1}{2^j}] \subset [0,1]$ which can be divided into two equal parts

$$\left[\frac{k}{2^j}, \frac{k}{2^j} + \frac{1}{2^{j+1}}\right] \qquad \text{and} \qquad \left]\frac{k}{2^j} + \frac{1}{2^{j+1}}, \frac{k+1}{2^j}\right]$$

where the function $h_{j,k}(x)$ takes, respectively, the values $2^{j/2}$ and $-2^{j/2}$. The diadic equality

$$\frac{1}{2^j} = \frac{1}{2^{j+1}} + \frac{1}{2^{j+1}} \tag{3.3}$$

is crucial in this process of 'multiresolution'. It is not difficult to show that the set

$$\{h_{j,k}\} \qquad j, k \in \mathbb{N},\ k < 2^j$$

together with the characteristic function $\chi_{[0,1]}$ forms an orthonormal basis of $L^2[0,1]$.

This follows from the possibility to write any real number as a binary number and from the subdivision (3.3).

A 'τ-adic' form of the same method is our subject here. One can write any real number in a 'τ-adic' form (2.10) and, due to the relation

$$\frac{1}{\tau^j} = \frac{1}{\tau^{j+1}} + \frac{1}{\tau^{j+2}} \tag{3.4}$$

we can subdivide the segment $[0,1]$ into smaller and smaller τ-adic subsegments. For example,

$$[0,1] = \left[0, \frac{1}{\tau}\right] \cup \left[\frac{1}{\tau}, 1\right]. \tag{3.5}$$

The length of the first one is $1/\tau$, while for the second one it is $1/\tau^2 = 1/\tau^3 + 1/\tau^4$:

$$\begin{aligned} \left[0, \frac{1}{\tau}\right] &= \left[0, \frac{1}{\tau^2}\right] \cup \left[\frac{1}{\tau^2}, \frac{1}{\tau}\right] \\ \left[\frac{1}{\tau}, 1\right] &= \left[\frac{1}{\tau}, \frac{1}{\tau} + \frac{1}{\tau^3}\right] \cup \left[\frac{1}{\tau} + \frac{1}{\tau^3}, \frac{1}{\tau} + \frac{1}{\tau^3} + \frac{1}{\tau^4}\right]. \end{aligned} \tag{3.6}$$

Continuing further in the same way, one arrives at the partition of $[0,1]$ with generic closed segment

$$\left[\frac{b}{\tau^j}, \frac{b+1}{\tau^j}\right] \tag{3.7}$$

where b is a τ-natural number satisfying the following two conditions:

(A) $0 \leqslant b \leqslant \tau^j - 1$;

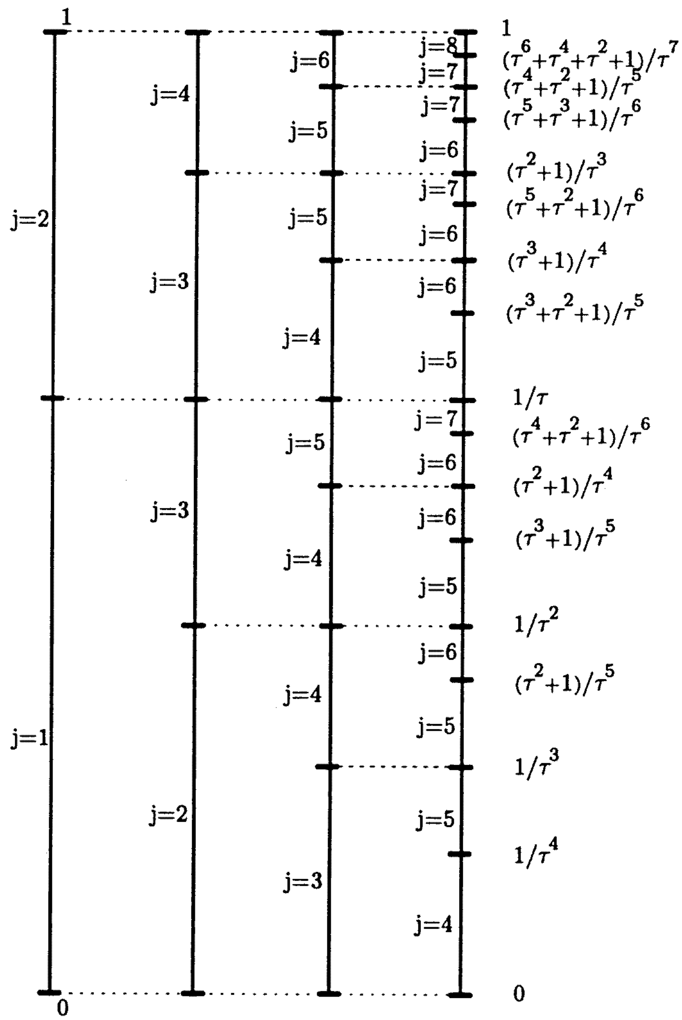

Figure 1. The first four steps of the τ-subdivision of the interval $[1, 0]$.

(B) the term 1 is not present in the τ-adic expansion of b, i.e. is a τ-integer multiplied by τ.

A few of the first successive partitions of $[0, 1]$ are shown in figure 1. Thus b has the general form

$$b = \tau^{j'} + k_{j'-1}\tau^{j'-1} + \cdots + k_1\tau \qquad \text{with } k_m \in \{0, 1\},\ k_m k_{m+1} = 0,\ j' < j. \tag{3.8}$$

Let us show that condition (A) holds for any b of the form (3.8). For that consider the obvious inequality

$$b \leqslant \tau^{j-1} + \tau^{j-3} + \cdots + k_1\tau \tag{3.9}$$

where $k_1 = 1$ for j even and $k_1 = 0$ for j odd. Thus we have

$$\begin{aligned} b &\leqslant \tau^{2p} + \cdots + \tau^2 = \tau^j - \tau \qquad j = 2p+1 \\ b &\leqslant \tau^{2p-1} + \cdots + \tau = \tau^j - 1 \qquad j = 2p. \end{aligned} \tag{3.10}$$

The segment (3.7) of length $1/\tau^j$ is the union of two segments of the next step of the subdivision, its two subsegments being of length $1/\tau^{j+1}$ and $1/\tau^{j+2}$:

$$\left[\frac{b}{\tau^j}, \frac{b+1}{\tau^j}\right] = \left[\frac{b}{\tau^j}, \frac{b}{\tau^j} + \frac{1}{\tau^{j+1}}\right] \cup \left[\frac{b}{\tau^j} + \frac{1}{\tau^{j+1}}, \frac{b+1}{\tau^j}\right]. \tag{3.11}$$

Clearly such a subdivision is always possible. Indeed each subsegment can be written as follows:

$$\left[\frac{b}{\tau^j}, \frac{b}{\tau^j} + \frac{1}{\tau^{j+1}}\right] = \left[\frac{b\tau}{\tau^{j+1}}, \frac{b\tau+1}{\tau^{j+1}}\right]$$

$$\left[\frac{b}{\tau^j} + \frac{1}{\tau^{j+1}}, \frac{b+1}{\tau^j}\right] = \left[\frac{b\tau^2+\tau}{\tau^{j+2}}, \frac{b\tau^2+\tau+1}{\tau^{j+2}}\right].$$

In both cases we obtain the form

$$\left[\frac{b'}{\tau^{j'}}, \frac{b'+1}{\tau^{j'}}\right]$$

where b' verifies the conditions (A) and (B):

(A) $0 \leqslant b' \leqslant \tau^{j'} - 1$ since $0 \leqslant b \leqslant \tau^j - 1$ implies $0 \leqslant b\tau \leqslant \tau^{j+1} - \tau \leqslant \tau^{j+1} - 1$ and $0 \leqslant b\tau^2 + \tau \leqslant \tau^{j+2} - \tau^2 + \tau = \tau^{j+2} - 1$.

(B) The term 1 is not present in the τ-adic expansion of b' because:

(B_a) 1 does not appear in the τ-expansion of b and *a fortiori* of $b\tau$.

(B_b) The lower power of τ in the τ-expansion of $b\tau^2$ is at least 3, and so the lower power of τ in the τ-expansion of $b\tau^2 + \tau$ is τ.

Let us now construct an orthonormal basis of $L^2[0, 1]$, analogous to the Haar basis, using the subdivision (3.11).

Let $h^\tau(x)$ be the function defined as follows:

$$h^\tau(x) = \begin{cases} \tau^{-1/2} & \text{for } x \in [0, \frac{1}{\tau}] \\ -\tau^{1/2} & \text{for } x \in]\frac{1}{\tau}, 1] \\ 0 & \text{for } x \notin [0, 1]. \end{cases} \tag{3.12}$$

One can verify directly that

$$\int_0^1 (h^\tau(x))^2 \,\mathrm{d}x = \frac{1}{\tau^2} + \tau\left(1 - \frac{1}{\tau}\right) = 1 \tag{3.13}$$

and that

$$\int_0^1 h^\tau(x) \,\mathrm{d}x = \tau^{-1/2} \cdot \frac{1}{\tau} - \tau^{1/2} \cdot \frac{1}{\tau^2} = 0. \tag{3.14}$$

Thus $h^\tau(x)$ is orthogonal to $\chi_{[0,1]}$. We therefore put

$$h^\tau_{j,b}(x) = \tau^{j/2} h^\tau(\tau^j x - b) \qquad h^\tau_{0,0} \equiv h^\tau \tag{3.15}$$

where $j \in \mathbb{N}$ and b is a positive τ-integer satisfying the conditions (3.8). The support of $h^\tau(x)$ is given in (3.7). More precisely

$$h^\tau_{j,b}(x) = \begin{cases} \tau^{j-1/2} & \text{for } \dfrac{b}{\tau^j} \leqslant x \leqslant \dfrac{b}{\tau^j} + \dfrac{1}{\tau^{j+1}} \\ -\tau^{j+1/2} & \text{for } \dfrac{b}{\tau^j} + \dfrac{1}{\tau^{j+1}} < x \leqslant \dfrac{b+1}{\tau^j} \\ 0 & \text{for } x \notin \left[\dfrac{b}{\tau^j}, \dfrac{b+1}{\tau^j}\right]. \end{cases} \tag{3.16}$$

The norm of $h^{\tau}_{j,b}(x)$ is 1,

$$\int_0^1 (h^{\tau}_{j,b}(x))^2\,\mathrm{d}x = \tau^{j-1}\cdot\frac{1}{\tau^{j+1}} + \tau^{j+1}\cdot\frac{1}{\tau^{j+2}} = 1. \tag{3.17}$$

It is orthogonal to $\chi_{[0,1]}$,

$$\int_0^1 h^{\tau}_{j,b}(x)\,\mathrm{d}x = \tau^{(j-1)/2}\cdot\frac{1}{\tau^{j+1}} - \tau^{(j+1)/2}\cdot\frac{1}{\tau^{j+2}} = 0 \tag{3.18}$$

and to all functions $h^{\tau}_{j,b}(x)$ with support which is either disjoint from that of (3.7), or containing completely (3.7), or contained completely in (3.7). There is no other possibility due to the process of subdivision of the segment which we have described in (3.11). Indeed, suppose that

$$\frac{b}{\tau^j} \leqslant \frac{b'}{\tau^{j'}} < \frac{b+1}{\tau^j} \qquad \text{with } j' \geqslant j \tag{3.19}$$

and let us demonstrate that

$$\frac{b'+1}{\tau^{j'}} \leqslant \frac{b+1}{\tau^j}. \tag{3.20}$$

The condition (3.19) reads $b\tau^{j'-j} \leqslant b' < b\tau^{j'-j} + \tau^{j'-j}$. The three numbers ordered by the above sequence of inequalities are τ-integers. The next one to b' in the ordered sequence of τ-integers is $b'+1$. The latter is at most equal to $b\tau^{j'-j} + \tau^{j'-j}$ from the strict inequality on the right and from $\tau^{j'-j} \geqslant 1$.

Now three cases arise. The first one is $j' = j$. Then $b \leqslant b' < b+1$ and thus $b = b'$: the segments are equal. Suppose next that $j' \geqslant j+1$. It follows that either

$$\frac{b'}{\tau^{j'}} \in \left[\frac{b}{\tau^j}, \frac{b}{\tau^j} + \frac{1}{\tau^{j+1}}\right) \qquad \text{or} \qquad \left[\frac{b}{\tau^j} + \frac{1}{\tau^{j+1}}, \frac{b+1}{\tau^j}\right).$$

In both cases a reasoning similar to the above one leads to the conclusion that, respectively, either

$$\left[\frac{b'}{\tau^{j'}}, \frac{b'+1}{\tau^{j'}}\right] \subset \left[\frac{b}{\tau^j}, \frac{b}{\tau^j} + \frac{1}{\tau^{j+1}}\right]$$

or

$$\left[\frac{b'}{\tau^{j'}}, \frac{b'+1}{\tau^{j'}}\right] \subset \left[\frac{b}{\tau^j} + \frac{1}{\tau^{j+1}}, \frac{b+1}{\tau^j}\right].$$

The result of these considerations is the property

$$\int_0^1 h^{\tau}_{j,b}(x)\,h^{\tau}_{j',b'}(x)\,\mathrm{d}x = \delta_{jj'}\delta_{bb'}. \tag{3.21}$$

Hence we have constructed an orthonormal set

$$\{\chi_0 \equiv \chi_{[0,1]},\ h^{\tau}_{j,b}(x) \,|\, j \in \mathbb{N},\ b \text{ is a } \tau\text{-integer verifying (3.8)}\}. \tag{3.22}$$

The set (3.22) is dense in $L^2[0,1]$. This is due to the fact that each open segment (c,d) in $[0,1]$ is a countable union of τ-adic segments.

First it is easily proved that each step function on $[0,1]$ is a pointwise limit of a sequence of linear combinations of characteristic functions

$$\chi_{[\frac{b}{\tau^j},\frac{b+1}{\tau^j})}(x) = \chi_{[0,1)}(\tau^j x - b) \equiv \chi_{j,b}(x) \tag{3.23}$$

where b satisfies (A) and (B). It follows from Lebesgue theory that the set (3.23) is dense in $L^2[0,1]$.

Then the $h^\tau_{j,b}$'s are obtained from the non-free set $\{\chi_0, \chi_{j,b}\}$ by the Gram–Schmidt orthogonalization:

$$h^\tau_{j,b} = \text{constant}\left[\chi_{j+1,\tau b} - \langle\chi_0|\chi_{j+1,\tau b}\rangle\chi_0 - \sum_{j',b'}\langle h^\tau_{j,b}|\chi_{j+1,\tau b}\rangle h^\tau_{j',b'}\right] \tag{3.24}$$

where the summation extends over all j' and b' such that

$$\left[\frac{b}{\tau^j}, \frac{b+1}{\tau^j}\right] \subset \left[\frac{b'}{\tau^{j'}}, \frac{b'+1}{\tau^{j'}}\right] \qquad \text{and} \qquad j' < j. \tag{3.25}$$

In return any characteristic function $\chi_{j,b}$ is a finite linear combination of functions in the free set $\{\chi_0, h^\tau_{j,b}\}$. Hence the latter is dense in $L^2[0,1]$ since the set of the former is.

The system (3.22) is an orthonormal basis of $L^2[0,1]$ which we call the *basis of* τ-*wavelets of Haar*.

4. τ-multiresolution analysis

The previous basis of τ-wavelets can easily be extended to $L^2(\mathbb{R})$. Here again we make extensive use of τ-integers. Let $\mathbb{Z}_\tau$ (resp. $\mathbb{Z}^+_\tau$) denote the set of τ-integers (resp. positive τ-integers). Then for any $j \in \mathbb{Z}$ and $b \in \tau\mathbb{Z}^+_\tau$ the set $\{h^\tau_{j,b}\}$ with

$$h^\tau_{j,b}(x) = \tau^{j/2}h^\tau(\tau^j x - b) \tag{4.1}$$

is an orthonormal basis of $L^2(\mathbb{R}^+)$. The proof is analogous to the one above for the case of $L^2[0,1]$. Note that we now relax condition (A) on b. The condition (B), i.e. $b \in \tau\mathbb{Z}^+_\tau$, requires b to be an 'even' τ-integer. The total support of the subset

$$\mathbb{S}^{\text{even}}_0 \equiv \{h^\tau_{0,b}\}_{b\in\tau\mathbb{Z}^+_\tau} \tag{4.2}$$

is less than the whole $\mathbb{R}^+$: intervals which are missing are the following

$$[b+1, b+\tau] \tag{4.3}$$

where $b \in \tau^2\mathbb{Z}^+_\tau$. These are the intervals of length $1/\tau$, unlike all the others

$$[b, b+1] \qquad b \in \tau\mathbb{Z}^+_\tau \tag{4.4}$$

which are of length 1. In other words, intervals (4.3) have 'τ-odd' lower bound. The set of 'τ-odd' integers is equal to

$$\mathbb{Z}^{\text{odd}}_\tau = \mathbb{Z}_\tau \backslash \tau\mathbb{Z}_\tau \qquad \text{i.e. } \mathbb{Z}_\tau = \tau\mathbb{Z}_\tau \cup \mathbb{Z}^{\text{odd}}_\tau. \tag{4.5}$$

We associate with the elements of $\mathbb{Z}^+_\tau$ the elements of the following Haar subset

$$\mathbb{S}^{\text{odd}}_0 := \{h^\tau_{1,b}\}_{b\in\tau\mathbb{Z}^{+\text{odd}}_\tau}. \tag{4.6}$$

It is now clear that the set

$$\mathbb{S}_0 := \mathbb{S}^{\text{even}}_0 \cup \mathbb{S}^{\text{odd}}_0 \tag{4.7}$$

has $\mathbb{R}^+$ as its total support and that it actually is the set of $\mathbb{Z}^+_\tau$-translated 'mother' wavelets, $h^\tau = h^\tau_{0,0}$. Finally the complete τ-Haar basis can be obtained from the set (4.2) through a simple reflection:

$$\tau\text{-Haar basis} = \{h^\tau_{j,b}, \overline{h^\tau_{j,b}}\}_{j\in\mathbb{Z}, b\in\tau\mathbb{Z}^+} \tag{4.8}$$

where $\overline{f(x)} := f(-x)$.

Other possibilities exist for extending the τ-Haar basis of $L^2(\mathbb{R}^+)$ to the whole $L^2(\mathbb{R})$. For example, one can define the τ-Haar basis on the negative side of the axis as the set

$$h^\tau_{j,-b-1} = \tau^{j/2} h^\tau(\tau^j x + b + 1) \qquad b \in \tau\mathbb{Z}^+,\ \ j \in \mathbb{Z}. \tag{4.9}$$

The τ-Haar basis

$$\{h^\tau_{j,b}, h^\tau_{j,-b-1}\}_{b\in\tau\mathbb{Z}^+,\ j\in\mathbb{Z}} \tag{4.10}$$

looks like a τ-translationally invariant set modulo dilation appropriate to the size of the support.

We can also change the sequence of negative τ-integers into another one which would be more 'self-similarly consistent' with $\mathbb{Z}^+_\tau$ but still quasiperiodically tiling the negative real axis with only two tiles of length 1 and $1/\tau$. This negative tiling should be chosen in order to get a set $\{h^\tau_{j,b}\}$ which is 'τ-translationally invariant' on the right modulo appropriate dilation.

The previous construction of the basis in $L^2(\mathbb{R})$ is better understood in the context of a multiresolution analysis based on the number τ instead of 2 as it has been formulated by Mallat [Ma89] and Meyer [Me92, Me93]. We adopt the method of Daubechies [D92] of construction of an orthonormal wavelet basis. Here we do it in the simplest case: the Haar basis. Subdividing the interval $[0, 1]$ into

$$[0, 1] = [0, \tfrac{1}{\tau}] \cup [\tfrac{1}{\tau}, 1] \tag{4.11}$$

leads to the functional equation

$$\varphi(x) = \varphi(\tau x) + \varphi(\tau^2 x - \tau) \tag{4.12}$$

for a function $\varphi(x) \in L^2(\mathbb{R})$. We also demand that

$$\int_{-\infty}^{+\infty} \varphi(x)\,\mathrm{d}x = 1 \tag{4.13}$$

and that the set

$$\{\varphi_{0,b}, \overline{\varphi_{0,b}}\}_{b\in\tau\mathbb{Z}^+_\tau} \cup \{\varphi_{1,b}, \overline{\varphi_{1,b}}\}_{b\in\tau\mathbb{Z}^{+\mathrm{odd}}_\tau} \tag{4.14}$$

where $\varphi_{j,b}(x) := \varphi(\tau^j x - b)$, be an orthonormal sequence in $L^2(\mathbb{R})$.

An obvious solution to (4.12) satisfying (4.13) and (4.14) is

$$\varphi(x) = \chi_{[0,1]}(x). \tag{4.15}$$

This solution is called the 'father wavelet'.

Let us denote by V_0 the closure of the linear span of (4.14). More generally, V_j, for $j \in \mathbb{Z}$, is defined from V_0 through the τ-scaling:

$$f(x) \in V_0 \Longleftrightarrow f(\tau^j x) \in V_j \tag{4.16}$$

for all $f \in L^2(\mathbb{R})$. Then

$$V_j \subset V_{j+1} \qquad \bigcap_{-\infty}^{+\infty} V_j = \{0\} \qquad \bigcup_{-\infty}^{+\infty} V_j \text{ is dense in } L^2(\mathbb{R}). \tag{4.17}$$

We now denote by W_j the orthogonal complement of V_j in V_{j+1},

$$V_{j+1} = V_j \oplus W_j. \tag{4.18}$$

Then there exists $\psi(x)$, the 'mother wavelet' in W_0, such that the set

$$\{\psi_{0,b}, \overline{\psi}_{0,b}\}_{b\in\tau\mathbb{Z}^+_\tau} \tag{4.19}$$

where $\psi_{j,b}(x) = \tau^{j/2}\psi(\tau^j x - b)$ is an orthonormal basis of W_0.

In our case $\psi(x) = h^\tau(x)$ as has been defined in (3.12). Then the set

$$\{\psi_{j,b}, \overline{\psi}_{j,b}\}_{b\in\tau\mathbb{Z}_\tau^+} \tag{4.20}$$

is an orthonormal basis of $L^2(\mathbb{R})$.

It is interesting to look at the functional equation satisfied by the Fourier transforms of φ and ψ because it is there that one finds the key of the Daubechies' construction. From (4.12) and (4.15) we have:

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\xi/2}\frac{\sin\xi/2}{\xi/2} &= \int_{-\infty}^{\infty}\varphi(t)\mathrm{e}^{-\mathrm{i}\xi t}\,\mathrm{d}t \\ &\equiv \hat{\varphi}(\xi) = \tau^{-1}\hat{\varphi}(\frac{\xi}{\tau}) + \tau^{-2}\mathrm{e}^{-\mathrm{i}\xi/\tau}\hat{\varphi}\left(\frac{\xi}{\tau^2}\right). \end{aligned} \tag{4.21}$$

From

$$\psi(x) = h^\tau(x) = \tau^{-1/2}\varphi(\tau x) - \tau^{1/2}\varphi(\tau^2 x - \tau) \tag{4.22}$$

we get

$$\psi(\xi) = \tau^{1/2}\left(-\hat{\varphi}(\xi) + \hat{\varphi}\left(\frac{\xi}{\tau}\right)\right). \tag{4.23}$$

Note the interesting limit formula obtained after iterating (4.21):

$$\mathrm{e}^{-\mathrm{i}\xi/2}\frac{\sin(\xi/2)}{\xi/2} = \frac{\tau^2+1}{\tau^2}\lim_{N\to\infty}\tau^{-N}\left(\sum_{0\leqslant\tau-\text{integer }b\leqslant\tau^{N-1}}\mathrm{e}^{-\mathrm{i}\xi b/\tau^{N-1}}\right). \tag{4.24}$$

This formula should be relevant in interpretation of diffraction pattern properties of pentacrystals [GS96]!

We finish the article with a few remarks concerning multiresolution formulae generalizing (4.12). In order to replace the step function by more regular ones, one can try higher-order scaling equations for the father wavelet. The steps next to (4.12) should have the form

$$\phi(x) = \sum_{j\geqslant 1}^{N}\sum_{k_j} c_{jk_j}\phi(\tau^j x - b_{k_j}) \qquad b_{k_j}\in\mathbb{Z}_\tau^+. \tag{4.25}$$

The solutions of (4.25) should be in L^2 and should satisfy (4.13) and some improved version of (4.14).

Equation (4.25) defines a solution ϕ from which a mother wavelet is deduced by using some recipes like orthogonality conditions [S89]. As a matter of fact we can propose the following equation as an immediate generalization of (4.12)

$$\phi(x) = c_0\phi(\tau x) + c_1\phi(\tau x - \tau) + c_2\phi(\tau^2 x - \tau^3 - \tau). \tag{4.26}$$

If ϕ has bounded support, the latter should be contained in $[0, \tau^2+1]$ which implies that the supports of the dilated–translated functions in (4.26) are respectively

$$[0, \tau + \tfrac{1}{\tau}] \qquad [1, \tau^2 + \tfrac{1}{\tau}] \qquad [\tau + \tfrac{1}{\tau}, \tau^2+1]. \tag{4.27}$$

The reason lying behind the choice of the interval $[0, \tau^2+1]$ and the three subsets (4.27) is that τ^2+1 is the first 'odd' τ-integer following 1 in the natural sequence of τ-integers, and only three subintervals with bounds in the sequence

$$\{0, \tfrac{1}{\tau}, 1, 1+\tfrac{1}{\tau^2}, \tau, \tau+\tfrac{1}{\tau}, \tau^2, \tau^2+\tfrac{1}{\tau}, \tau^2+1\}$$

are appropriate to the first multiresolution step.

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